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Kinetic terms in warped compactifications

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Abstract: We develop formalism for computing the kinetic terms of 4d fields in string compactifications, particularly with warping. With the help of the Hamiltonian approach, we identify a gauge dependent inner product on the compactification manifold which depends on the warp factor. It is shown that kinetic terms are associated to the minimum value of the inner product over each gauge orbit. We work out the kinetic term for the complex modulus of a deformed conifold with flux, i.e. the Klebanov-Strassler solution embedded in a compact Calabi-Yau manifold. Earlier results of a power-like divergence are confirmed qualitatively (the kinetic term does contain the main effect of warping) but not quantitatively (the correct results differ by an order one coefficient).

Keywords: Flux compactifications, Superstring Vacua

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1 Introduction

One of the central problems of string/M theory is to find consistent compactifications and work out their four dimensional low energy descriptions. Most work starts with the 10d or 11d supergravity theory and does Kaluza-Klein reduction, and then considers stringy and quantum effects as corrections depending on small parameters. We can refer to a regime in which such an expansion is good as a "supergravity limit." Using duality, many strong coupling limits can be reformulated as other weakly coupled supergravity limits. But on general grounds one expects other "order one coupling" regimes to exist, and there has been much effort to understand them, by summing instantons, using holomorphy, interpolating between different weakly coupled regimes, etc.

While this is an important goal, the supergravity limits already realize a great deal of interesting physics, and could be better understood. Indeed, our experience has been that a key to the general problem has been to identify mathematical structures present in the supergravity limit, which persist in the general case. This was the case for mirror symmetry, both closed string (variation of Hodge structure) and open string (categorical and A_∞ structure, stability conditions). Thus, our goals include both developing practical calculational techniques, and to find such structures.

In the present work, we focus on the problem of computing kinetic terms. Our immediate motivation was the study of type IIb flux compactification carried out in [1, 2] along lines initiated in [3]. These are warped compactifications, and Kaluza-Klein reduction in such backgrounds is subtle, with various incorrect and incomplete results in the literature. One reason for this is that, in following the standard approach of substituting the Kaluza-Klein ansatz into the Lagrangian, one finds that one needs "compensator" fields [4, 5], which are difficult to solve for explicitly, and do not (at least to us) suggest any clear physical or mathematical intuition for the results.

As it turns out, a fairly direct route to the kinetic terms is to derive them in a Hamiltonian framework. The reason is that the system has constraints associated to gauge redundancies, while the physical degrees of freedom become manifest in the Hamiltonian formulation. While this does not completely eliminate the need to discuss compensators, it does provide a much clearer picture of why they arise and how to deal with them.

Perhaps the simplest way to explain the main point is to realize that the kinetic terms for metric moduli originate from a metric on the space of metrics, but the usual expression for this metric is gauge dependent. A mathematically natural [6] and physically correct [7] way to fix this ambiguity is to require that the metric fluctuations be orthogonal to gauge transformations. However, when one says "orthogonal," one has implicitly used the ten-dimensional metric, in a way which sees the warp factor. This is the point at which warping changes the usual discussion.

1.1 General problem

We consider a D-dimensional theory of gravity coupled to matter, e.g. a supergravity. A vacuum solution is a solution of the equations of motion which at long distances "looks like" a d-dimensional space M with maximal symmetry, i.e. Minkowski space, AdS or dS. In general it will be a product or warped product of M with an $n = D - d$-dimensional compactification space (or internal space) X, possibly with other nonzero fields consistent with maximal symmetry (i.e. scalars, components of vector fields in X, etc.).

We use x^μ and y^i to denote coordinates on M and X respectively. For definiteness we will sometimes take $D = 10$ and $d = 4$, but our considerations will not depend on this.

Suppose there is a family of vacuum solutions of the D-dimensional equations of motion, with parameters u^I. Thus we can write $g_{MN}(y; u)$, $\phi(y; u)$, and so forth. To analyze the dynamics of these moduli u^I, we might try to find a family of "approximate solutions" of the equations of motion, obtained by taking the parameters to slowly vary on M [7]:

$$g_{MN}(y, u(x)). \tag{1.1}$$

The kinetic terms are then the terms in the d-dimensional effective Lagrangian of the form

$$\int d^d x \, \sqrt{g} g^{\mu\nu} \, G_{IJ}(u) \, \partial_\mu u^I \partial_\nu u^J \,, \tag{1.2}$$

obtained by substituting Eq. (1.1) into the D-dimensional action, integrating over X and identifying these terms.[1] Note that to compute Eq. (1.2), we need to allow "off-shell" $u(x)$ (i.e. $\partial^2 u \neq 0$).

However, this direct approach can become complicated. The first sign of this is that in general, the ansatz Eq. (1.1) does not solve the ten-dimensional equations of motion, even when $u(x)$ solves the four-dimensional massless field equations. One may need a more general ansatz depending on derivatives $\partial u, \partial^2 u$, etc. Further subtleties arise from gauge invariance. We will see how this happens and its consequences in examples.

1.2 Summary

We start in section 2 with the example of Yang-Mills theory, which is used to illustrate in a simple setup many of the subsequent points. Then in section 3 we construct a Riemannian metric on the space of metrics, with the help of the Hamiltonian of General Relativity. This metric is used in section 4 to construct kinetic terms arising from 10d (warped) backgrounds preserving 4d maximal symmetry. We prove that metric fluctuations should be orthogonal to gauge transformations associated to the full warped metric. This turns out to be equivalent to minimizing the value of their inner product over each gauge orbit.

In section 5, the previous formalism is applied to string compactifications. We first discuss the case of a Calabi-Yau manifold, where the metric for complex and Kähler moduli is recovered. The harmonic gauge choice generally considered in the literature is identified as a dynamical constraint. Next the more interesting case of conformal Calabi-Yau compactifications is analyzed; these correspond to type IIb supergravities with BPS branes and fluxes. Compensating fields are identified with Lagrange multipliers of the Hamiltonian. Their role is to set metric fluctuations into harmonic gauge with respect to the full warped metric. We find a fairly simple expression for the field space metric in terms of warped metric fluctuations. Upon rewriting this in terms of the underlying Calabi-Yau moduli we verify the expression recently found in [1].

Finally, in section 6 we compute the metric for the complex modulus S of the warped deformed conifold. We find a power-like divergence $|S|^{-4/3}$ that agrees with the analysis done in [2]. Both results differ, however, by a numerical coefficient. The reason for this is that before it was not known how to construct fluctuations orthogonal to gauge transformations.

[1] The correct action may require a boundary term to cancel boundary terms in the variation, for example the Gibbons-Hawking-York term in general relativity.

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2 Yang-Mills theory

We start with the simple case of a U(1) field A_M with field strength F_{MN}. We suppose that there are a family of solutions of

$$D^i F_{ij} = 0$$

on X, parameterized by coordinates u^I. For example, if X is a torus, every flat connection is a solution, and the u^I might be the holonomy associated to a basis of $H^1(X, \mathbb{Z})$.

We take as the ten-dimensional action

$$S = \ldots - \frac{1}{4} \int d^4x \sqrt{g_4} \int d^6y \sqrt{g_6} g^{MN} g^{PQ} F_{MP} F_{NQ}. \tag{2.1}$$

Naively we then set $A_\mu = 0$ and write

$$F_{\mu i} = \partial_\mu A_i(y; u(x)) - \partial_i A_\mu(y; u(x)) = \frac{\partial A_i}{\partial u^I} \partial_\mu u^I,$$

and substitute this into the action, to obtain Eq. (1.2) with

$$G_{IJ} = \int d^6y \sqrt{g_6} g^{ij} \frac{\partial A_i}{\partial u^I} \frac{\partial A_j}{\partial u^J}. \tag{2.2}$$

However, on reflection, there must be a subtlety in this procedure. In defining our moduli space of solutions $A_i(y; u)$, nowhere did we specify a gauge for A_i. Two solutions which are related by gauge transformations on X,

$$\delta A_i = \partial_i \epsilon,$$

are equally good from the point of view of X. On the other hand, the expression Eq. (2.2) is not gauge invariant, so the kinetic terms will depend on which of the gauge equivalent solutions we take. Since Eq. (2.1) was gauge invariant in ten dimensions, we must have made an error.

The error was the assumption that $A_\mu = 0$ for all of these solutions. Let us look at the ten dimensional equations of motion. These can be written as

$$0 = D^\mu F_{\mu\nu} + D^i F_{i\nu}; \qquad 0 = D^\mu F_{\mu j} + D^i F_{ij}. \tag{2.3}$$

We substitute the ansatz $A_i(y; u(x))$ and require that there is no four-dimensional gauge field, $F_{\mu\nu} = 0$. This sets

$$A_\mu(x, y) = \Omega(y) \partial_\mu f(x)$$

where $\Omega(y)$ and $f(x)$ are still undetermined functions.

To find A_μ, we use the first equation of motion, which becomes $0 = \partial^i F_{i\nu}$, i.e.

$$\partial^i \partial_\nu A_i = \partial^i \partial_i A_\nu. \tag{2.4}$$

In general, the left hand side is nonzero, so we will have $A_\nu \neq 0$. However a simple way to make the left hand side zero is to require

$$0 = \partial^i \frac{\partial A_i}{\partial u^I}, \tag{2.5}$$

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i.e. the fluctuations are taken in harmonic gauge. More generally, solving Eq. (2.4) produces an A_ν which is the parameter of the "compensating gauge transformation",

$$A_\mu(x,y) = \Omega_I(y)\partial_\mu u^I(x)\ , \qquad \partial^i\partial_i\Omega_I = \partial^i\,\frac{\partial A_i}{\partial u^I}\,. \tag{2.6}$$

Defining

$$\delta_I A_i := \frac{\partial A_i}{\partial u^I} - \partial_i\Omega_I\,, \tag{2.7}$$

we see that the effect of Ω is to put $\delta_I A_i$ back into harmonic gauge.

In general, it is hard to explicitly solve Eq. (2.4) for the compensator field A_ν. However, to compute the kinetic term, we do not need to do this, rather we just need to impose the condition Eq. (2.5).

2.1 Metric for Yang-Mills connections

One can straightforwardly generalize the above to nonabelian gauge fields. There is also a simple geometric interpretation of the final result, which leads immediately to the metric both for Yang-Mills and for gravitational configurations.

Note that Eq. (2.5) is the condition that the variation $\delta_I A$ is orthogonal in the metric Eq. (2.2) to all the gauge directions. This is a natural mathematical condition and leads to a unique definition of the metric [6].

Let $\mathcal{A}$ be the set of possible (smooth) gauge potentials on $\mathbb{R}^3$, and $\mathcal{G}$ be the group of all gauge transformations over $\mathbb{R}^3$. The four-dimensional physical configuration space is then the quotient (or orbit space) $\mathcal{C} \equiv \mathcal{A}/\mathcal{G}$.

Given a metric g_{ij} on $\mathbb{R}^3$, there is a natural metric on $T\mathcal{A}$,

$$(\dot{A},\dot{A}) = \int d^3x\ \sqrt{g}g^{ij}\ \operatorname{tr}\left(\dot{A}_i(x)\dot{A}_j(x)\right). \tag{2.8}$$

Given a path $c(t)$ in $\mathcal{C}$, we would like to define a natural Riemannian metric H on $\mathcal{C}$, which can be used in a particle action as [7]

$$S[c] = \int dt\,\frac{1}{2}\,H(\dot{c},\dot{c})\,. \tag{2.9}$$

Since actually one works with paths $A_i(t) \in \mathcal{A}$, the basic requirement is that $S[c]$ should be independent of the way $c(t)$ is lifted to $\mathcal{A}$. This can be accomplished by projecting the tangent vector $\dot{A}_i(t)$ on the subspace orthogonal to gauge transformations in the metric Eq. (2.8). Thus, let Π_i be this projection,

$$\Pi_i(\dot{A}) := \dot{A}_i - D_i(1/D^2)D_j\dot{A}_j\ , \qquad \Pi_i(D_k\lambda) = 0\,. \tag{2.10}$$

The natural metric on $\mathcal{C}$ is then

$$H(\dot{c},\dot{c}) = \int d^3x\,\operatorname{tr}\big(\Pi_i(\dot{A})\,\Pi_i(\dot{A})\big)\,. \tag{2.11}$$

From a physics point of view, $\Pi_i(\dot{A})$ is the electric field F_{0i} after eliminating A_0 by using the Gauss law. Equivalently, the projector is given by the nonabelian version of the zero mode Eq. (2.7) after solving for the compensator Ω. Substituting into the E^2 terms of the Yang-Mills action, one obtains Eq. (2.9).

There are several other formulations of the same result. One is to regard the configuration space $\mathcal{A}$ as a $\mathcal{G}$-bundle over the space of gauge orbits. The projection Eq. (2.10) then defines a preferred notion of "parallel transport" on this bundle, making Eq. (2.8) unambiguous. The metric Eq. (2.9) is then gauge invariant, in the sense that it is derived from a gauge invariant notion of parallel transport.

Another formulation is to note that, since the metric Eq. (2.8) is positive definite, evaluating it with the gauge directions projected out is the same as evaluating it on the gauge representative which *minimizes* its value.

2.2 Relation to Hamiltonian formulation

A slightly different way of reducing to gauge invariant variables is to go to the Hamiltonian formulation. We recall that, since the time derivatives $\partial_0 A_0$ do not appear in the action, the A_0 component of the vector potential plays the role of a Lagrange multiplier, which is conjugate to the Gauss law,

$$S = \ldots + \int A_0 D_i E^i.$$

One can then enforce the Gauss law as a constraint on the initial data (A_i, E^i), which is preserved under Hamiltonian evolution.

This is a particular example of "symplectic reduction" with respect to a symmetry group G. Starting with a phase space M with a symplectic structure $\omega(u,v)$, one identifies "moment maps" μ which are "Hamiltonians" generating the infinitesimal action of G. One can then show that the reduced phase space

$$\{x \in M : \mu(x) = 0\}/G$$

carries a symplectic structure.

In the Yang-Mills example, $G = \mathcal{G}$, and M is the direct product of the space $\mathcal{A}$ of connections $A_i(x)$ with the space of electric field strengths $E^i(x)$. It carries the symplectic structure

$$\omega(A,E) = \int d^3x \; \mathrm{tr} \left(A_i(x) E^i(x) \right).$$

The moment maps for $\mathcal{G}$ are then $\mu = D_i E^i$. Thus, the Gauss law constraint is the natural partner of the gauge condition in this construction as well. Since the E^2 terms in the Hamiltonian are gauge invariant, they are single valued on the reduced phase space, resulting in the same metric Eq. (2.9).

Physically, we can use this formulation by considering a configuration in which the moduli u^I are linearly varying with time. The metric is then the energy density of this configuration, and the Hamiltonian framework provides a direct way to compute this. Since the phase space does not contain time-like components of vector potentials, there is no possibility for a "compensator field" A_0 to enter; rather the mixed equations of motion such as Eq. (2.3) are solved implicitly in this framework.

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In general, the result of this prescription will depend on the initial choice of symplectic structure on field space. However in field theory there is usually a unique local candidate for this structure.

3 General relativity

In this section we consider the problem of constructing a natural Riemannian metric on the space of metrics. This will be done by using the Hamiltonian formulation of general relativity, which is well-suited for extracting the kinetic terms in a general case. At the end of the section we present a simple example where the kinetic terms are obtained via the usual Lagrangian approach, so that both perspectives may be compared.

3.1 Metric on the space of metrics

The problem may be formulated as follows. Consider a D-dimensional manifold equiped with a metric $g_{MN}(x)$, $M, N = 0, \ldots, D$. In many cases of interest the metric satisfies certain background equations of motion. For example, in pure Einstein gravity it is Ricci flat. However these equations depend on the theory, and thus we will not make use of them in this section.

We identify a time coordinate $t = x^0$; then Σ denotes the space-like surface $t = 0$ and h_{MN} is the pull-back of g_{MN} to Σ. Let $\mathcal{A}$ be the set of all such possible Riemannian metrics h_{MN}, and $\mathcal{G}$ the corresponding diffeomorphisms. Our aim is to identify a Riemannian metric H on $\mathcal{A}/\mathcal{G}$ and then for each path $c(t) \in \mathcal{A}/\mathcal{G}$ introduce a natural action

$$S[c] = \int dt \, \frac{1}{2} H(\dot{c}, \dot{c}) \, . \tag{3.1}$$

Following the previous discussion it will now be shown how this arises from the Hamiltonian formulation for GR [9, 10].

One starts by prescribing initial value conditions on a $D - 1$ dimensional space-like surface Σ_0, with metric h_{MN}. Denoting its unit normal vector by n_N,

$$h_{MN} = g_{MN} + n_M n_N \, . \tag{3.2}$$

The equations of motion produce the time evolution $\Sigma_0 \to \Sigma_t$, and the physical degrees of freedom are h_{MN} and not g_{MN}. The remaining components, denoted by η_N, are determined in terms of the "dual" vector t^M, which satisfies

$$(g_{tt})^{1/2} = -g_{MN} t^M n^N \, .$$

Recall the gauge choice $t = x^0$; also, $g_{tt} = -g_{00} > 0$. Then,

$$\eta^N = h^{NM} \eta_M = t^N - (g_{tt})^{1/2} n^N \, .$$

The geometrical interpretation is that the time evolution $\Sigma_0 \to \Sigma_t$ given by the vector field t^N can be decomposed into a normal direction n^N plus a tangential shift η^N. The dynamics is encoded in the extrinsic curvature,

$$K_{MN} := \frac{1}{2} \mathcal{L}_n h_{MN} = \frac{1}{2} (g^{tt})^{1/2} (\dot{h}_{MN} - D_N \eta_M - D_M \eta_N) \, , \tag{3.3}$$

where D_N is the covariant derivative on Σ, compatible with h_{MN}. The lagrangian density takes the form

$$\mathcal{L}_G = \sqrt{-g_D}\Big(R^{(D-1)} + K_{MN}K^{MN} - K^2\Big)\,. \tag{3.4}$$

In terms of these variables, the canonical momentum reads

$$\pi^{MN} = \frac{\partial \mathcal{L}_G}{\partial \dot{h}_{MN}} = h^{1/2}(K^{MN} - h^{MN}K)\,, \tag{3.5}$$

from which we obtain the Hamiltonian density,

$$\mathcal{H}_G = \sqrt{-g_D}\bigg(-R^{(D-1)} + h^{-1}\pi^{MN}\pi_{MN} - \frac{1}{D-2}h^{-1}\pi^2\bigg) - 2h^{1/2}\eta_N D_M(h^{-1/2}\pi^{MN}) \tag{3.6}$$

The shift vectors η^N are Lagrange multipliers which enforce the constraints

$$D_N(h^{-1/2}\pi^{NM}) = 0\,. \tag{3.7}$$

After satisfying this we can set $\eta^N = 0$, as usual in constrained Hamiltonian systems.

The Riemannian metric on $\mathcal{A}/\mathcal{G}$ corresponds to the kinetic term of the Hamiltonian density. Given a path $c_{MN}(t) \in \mathcal{A}/\mathcal{G}$ we introduce a lift $h_{MN}(t)$ to $\mathcal{A}$; to the tangent vector $\dot{h}_{MN}$ we associate the "projection" $\pi^{MN}(\dot{h})$ defined in Eq. (3.5). The metric on the space of metrics becomes

$$H(\dot{c},\,\dot{c}) = \int d^{D-1}x\,\sqrt{-g_D}\bigg(h^{-1}\pi^{MN}\pi_{MN} - \frac{1}{D-2}h^{-1}\pi^2\bigg) \tag{3.8}$$

and the action is given by Eq. (3.1). The constraint Eq. (3.7) implies that $\pi^{MN}(\dot{h})$ is orthogonal to gauge transformations,

$$H(\mathcal{L}_v\pi,\,\pi) = 0\,.$$

Actually, π^{MN} itself is a projector $\mathcal{A} \to \mathcal{A}/\mathcal{G}$:

$$\pi^{MN}(\mathcal{L}_v\dot{h}) = 0\,.$$

The proof is analogous to the YM case Eq. (2.10), and is based on eliminating the Lagrange multipliers η_N. We conclude that the Hamiltonian approach to GR yields a natural Riemannian metric Eq. (3.8) on $\mathcal{A}/\mathcal{G}$.

3.2 Unwarped solutions

In simple cases it is still possible to compute kinetic terms using the Lagrangian formulation, as we now discuss in an example. Consider a family of six dimensional Ricci-flat manifolds X with metric $g_{ij}(y;u)$. Examples are Calabi-Yau manifolds, with u^I parametrizing complex and Kähler moduli. The ten dimensional background is taken to be the unwarped product $M \times X$ with metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + g_{ij}(y;u)dy^i dy^j\,. \tag{3.9}$$

Promoting the moduli to fields $u^I(x)$ fibers X over M, but only through the implicit dependence of the moduli on the space-time coordinates. As in the Maxwell case, just replacing $u \to u(x)$ into Eq. (3.9) doesn't give a consistent D-dimensional solution. To satisfy $G_{MN} = 0$, we consider the following ansatz including a compensating field B_i:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + 2B_{Ij}(y)\partial_\mu u^I dy^j dx^\mu + g_{ij}(y; u(x))dy^i dy^j \,. \tag{3.10}$$

It has been pointed out [5] that an extra compensator term of the form $K_I(y)\partial_\mu\partial_\nu u^I\, dx^\mu dx^\nu$ may also be needed. However, we will show that B_{Ij} is only defined modulo a total derivative term, which can be used to set $K_I = 0$.

The components of the Einstein tensor, up to two space-time derivatives, read

$$G_{\mu\nu} = (\partial_\mu\partial_\nu u^I - g_{\mu\nu}\Box u^I)\left[-\frac{1}{2}\frac{\partial g}{\partial u^I} + \nabla^j B_{Ij}\right] \tag{3.11}$$

$$G_{\mu i} = \frac{1}{2}\partial_\mu u^I \nabla^j\left(\nabla_i B_{Ij} - \nabla_j B_{Ii} + \frac{\partial g_{ij}}{\partial u^I} - g_{ij}\frac{\partial g}{\partial u^I}\right) \tag{3.12}$$

$$G_{ij} = -\frac{1}{2}\Box u^I\left[\frac{\partial g_{ij}}{\partial u^I} - \nabla_i B_j - \nabla_j B_i\right] , \tag{3.13}$$

where the trace part is

$$\frac{\partial g}{\partial u^I} := g^{ij}\frac{\partial g_{ij}}{\partial u^I} \,.$$

A consistent ten dimensional solution requires $G_{\mu i} = 0$, which fixes B_{Ij}, up to a total derivative $\partial_j K_I$. Then we have to require that $G_{\mu\nu} = 0$, off-shell for $u(x)$, which determines the previous function K_I:

$$\nabla^j B_{Ij} = \frac{1}{2}\frac{\partial g}{\partial u^I} \,. \tag{3.14}$$

Using Eq. (3.14) to eliminate $\partial_I g$, Eq. (3.12) can be rewritten more suggestively as

$$\nabla^i\left[\frac{\partial g_{ij}}{\partial u^I} - \nabla_i B_j - \nabla_j B_i\right] = 0 \,. \tag{3.15}$$

Plugging these results in the Einstein-Hilbert action, the action up to two space-time derivatives is of the form Eq. (1.2), with field space metric

$$G_{IJ}(u) = \frac{1}{4}\int d^6y\sqrt{g_6}\, g^{ij} g^{kl}\, \delta_I g_{ik}\, \delta_J g_{jl} \tag{3.16}$$

where

$$\delta_I g_{ij} := \frac{\partial g_{ij}}{\partial u^I} - \nabla_i B_j - \nabla_i B_j \,. \tag{3.17}$$

The role of the ten dimensional constraints is to set $\delta_I g_{ij}$ in the transverse traceless gauge,

$$\nabla^i\, \delta_I g_{ij} = 0\ ,\ g^{ij}\delta_I g_{ij} = 0 \,. \tag{3.18}$$

This example shows how the metric compensators repackage into a "physical" zero mode $\delta_I g_{ij}$ which is orthogonal to diffeomorphism transformations. Their effect can be simply summarized in the requirement that the zero mode has to be in the transverse traceless gauge. The upshot from this example is that harmonic gauge is not a choice, but rather a dynamical constraint.

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4 Kinetic terms in general compactifications

The most general D-dimensional metric consistent with d-dimensional maximal symmetry is

$$ds^2 = e^{2A(y;\,u)}\,\hat{g}_{\mu\nu}(x)dx^\mu dx^\nu + g_{ij}(y;u)dy^i dy^j\,. \tag{4.1}$$

This is a warped product of a maximally symmetric space M with metric $\hat{g}_{\mu\nu}$ and an arbitrary compactification manifold X with metric g_{ij}. The internal manifold depends on parameters u^I and the aim is to find their kinetic terms. This applies to all supergravity compactifications preserving 4d maximal symmetry.

We will assume here that g_{ij} does not have exact isometries, as is the case in CY manifolds. This simplifies the analysis, since there are no gauge fields coming from the off-diagonal fluctuations $\delta g_{\mu m}$. There is a mass gap and $\delta g_{\mu m}$ are associated to massive spin 1 fields, which we choose not to excite. In a more complete treatment, one should describe how such fields combine with the graviton modes (and scalars from the internal manifold) to yield massive spin 2 degrees of freedom.

The situation is a particular case of that discussed in the previous section, where the path $c(t)$ corresponds to promoting u^I to spacetime fields. Since the 4d part $\hat{g}_{\mu\nu}$ is fixed, the metric on the space of metrics should now reduce to a metric on the parameter space $\{u^I\}$. We will not assume that g_{ij} is Ricci-flat; rather, it satisfies certain background equations of motion (for instance, including fluxes). The advantage of the Hamiltonian approach is that the identification of the kinetic term does not require analyzing such equations.

Once the u^I are allowed to fluctuate, we have to include compensators B_{Ij},

$$ds^2 = e^{2A(y;\,u)}\left(\hat{g}_{\mu\nu}(x)dx^\mu dx^\nu + 2B_{Ij}(y)\partial_\mu u^I\, dx^\mu dy^j\right) + g_{ij}(y;u)dy^i dy^j\,. \tag{4.2}$$

In the Lagrangian approach, the compensators are fixed by solving the equations of motion at linear order in velocities. Once this is done, the kinetic terms may be extracted from the equations which are quadratic in space-time derivatives.

Here the system will be analyzed from a Hamiltonian point of view; for simplicity, we take $\partial_\mu u^I = \delta^0_\mu \dot{u}^I$.[2] The kinetic term for the moduli $u^I(t)$ is obtained by plugging the corresponding time-dependent metric $\dot{h}_{MN} = \dot{u}^I(\partial h_{MN}/\partial u^I)$ in Eq. (3.8). In the linearized approximation the extrinsic curvature K_{MN} and canonical momentum π_{MN} are both proportional to $\dot{u}^I$, so we can write

$$K_{MN} = \frac{1}{2}\,(g^{tt})^{1/2}\,\dot{u}^I\,\delta_I h_{MN}\ ,\ \ h^{-1/2}\pi_{MN} = \frac{1}{2}\,(g^{tt})^{1/2}\,\dot{u}^I\,\delta_I \pi_{MN} \tag{4.3}$$

and the factors of $(g^{tt})^{1/2}/2$ have been extracted for later convenience. The coefficients $\delta_I h_{MN}$ and $\delta_I \pi_{MN}$ are given by

$$\delta_I h^{MN} = \frac{\partial h^{MN}}{\partial u^I} - D^M \eta_I^N - D^N \eta_I^M \tag{4.4}$$

$$\delta_I \pi^{MN} = \delta_I h^{MN} - h^{MN}\, h^{PQ}\, \delta_I h_{PQ} \tag{4.5}$$

[2] Recall that the difference between g_{MN} and h_{MN} is that the latter only includes space-like components.

where we have expanded $\eta^N = \dot{u}^I \, \eta_I^N$.

The relation between the Lagrangian and Hamiltonian approach is that the compensators coincide with the Lagrange multipliers η_M,

$$\eta_{I\mu} = 0 \; , \; \eta_{Ij} = e^{2A} \, B_{Ij}(y) \, . \tag{4.6}$$

The advantage of the Hamiltonian formulation is that they appear explicitly as nonpropagating fields, whose only role is to impose the constraints

$$D^N \big((g^{tt})^{1/2} \delta_I \pi_{MN} \big) = 0 \, , \tag{4.7}$$

which imply that the physical variations are orthogonal to gauge transformations. We remind the reader that D_N is the covariant derivative compatible with the space-like metric h_{MN}. The kinetic term derived from the Hamiltonian Eq. (3.8) reads

$$\begin{aligned} H &= \frac{1}{4} \, \dot{u}^I \, \dot{u}^J \, \left(\int d^{D-1}x \, \sqrt{-g_D} \, g^{tt} \left[\delta_I \pi_{MN} \, \delta_J \pi^{MN} - \frac{1}{D-2} \, \delta_I \pi \, \delta_J \pi \right] \right) \\ &= \frac{1}{4} \, \dot{u}^I \, \dot{u}^J \, \left(\int d^{D-1}x \, \sqrt{-g_D} \, g^{tt} \, \delta_I \pi_{MN} \, \delta_J h^{MN} \right) . \end{aligned} \tag{4.8}$$

This is the gravitational analog of the kinetic term $p \, \dot{q}$ in particle mechanics.

Let us now prove that Eq. (4.7) is equivalent to *minimizing* the inner product over each gauge orbit. Under a gauge transformation

$$\delta_I h^{MN} \to \delta_I h^{MN} - D^N v_I^M - D^M v_I^N \, ,$$

the change in the inner product Eq. (4.8) is

$$- \, 2 \int d^{D-1} \sqrt{-g_D} \, g^{tt} \, v_I^M \, D^N \left[(g^{tt})^{1/2} \left(\delta_J \pi_{MN} + \mathcal{L}_v \, \delta_J \pi_{MN} \right) \right] \, . \tag{4.9}$$

Demanding that the gauge parameter minimizes this expression, we find

$$D^N \left[(g^{tt})^{1/2} \left(\delta_J \pi_{MN} + \mathcal{L}_v \, \delta_J \pi_{MN} \right) \right] = 0 \, , \tag{4.10}$$

thus reproducing the prescription given in Eq. (4.7).

4.1 Four dimensional expression

To compactify over the internal manifold one would in principle need to know the warp factor and then extract the variation $\partial_I A$. These are complicated functions determined by the background equations of motion. But interestingly, the constraints Eq. (3.7) fix $\delta_I A$ in terms of $g^{ij} \, \delta_I g_{ij}$: from

$$0 = D^\mu (\delta_I \pi_{\mu\nu}) = -\partial_\nu \big(2 \, e^{-2A} \, \delta_I e^{2A} + g^{ij} \, \delta_I g_{ij} \big) \, ,$$

we obtain

$$\delta_I e^{2A} = -\frac{1}{2} \, e^{2A} \, g^{ij} \, \delta_I g_{ij} \, . \tag{4.11}$$

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This implies that $\delta\pi_{\mu\nu} = 0$, while the warp factor variation may be eliminated from $\delta\pi_{ij}$ yielding

$$\delta_I \pi_{ij} = \delta_I g_{ij} + \frac{1}{d-2} g_{ij}\, g^{kl} \delta_I g_{kl} \,. \tag{4.12}$$

The internal part of the constraint sets

$$D^N (e^{-A} \delta_I \pi_{Nj}) = 0 \,, \tag{4.13}$$

where e^{-A} comes from $(g_{tt})^{-1/2}$, and it is important to remember that the connection is defined with respect to the full warped metric. To rewrite this in terms of 6d variables, notice that

$$D^\mu (e^{-A} \delta_I \pi_{\mu j}) = 3\, e^{-A}\, \partial^k A\, \delta_I \pi_{kj}$$

where we used the fact that $\pi_{\mu\nu} = 0$ and $h^{\mu\nu}\Gamma^k_{\mu\nu} = -3\partial^k A$. Then (4.13) becomes

$$g^{ij} \nabla_i (e^{2A} \delta_I \pi_{jk}) = 0 \,. \tag{4.14}$$

With these results, the general formula for the kinetic terms is[3]

$$S_{kin} = \frac{1}{2} \int d^d x\, \sqrt{-\hat{g}_d}\, \hat{g}^{tt}\, \dot{u}^I \dot{u}^J\, G_{IJ}(u) \tag{4.15}$$

with

$$G_{IJ}(u) = \frac{1}{4} \int d^{D-d} y \sqrt{g_{D-d}}\, e^{2A}\, \delta_I g_{ij}\, \delta_J \pi^{ij} \,. \tag{4.16}$$

The warp factor dependence comes from $\sqrt{-g_d}\, g^{tt} = \sqrt{-\hat{g}_d}\, \hat{g}^{tt}\, e^{2A}$. From this expression it becomes clear that Eq. (4.14) is simply the condition that the physical variation $\delta_I \pi_{ij}$ is orthogonal to gauge transformations. The effects of the compensators are summarized in this prescription.

4.2 Effect of compensators

The Hamiltonian approach shows that the effect of the compensators is to make the metric fluctuations orthogonal to gauge transformations. In general it is simpler to compute the "naive" zero modes just by taking derivatives $\frac{\partial g_{ij}}{\partial u^I}$. The metric associated to these fluctuations is

$$G^0_{IJ} = \frac{1}{4} \int d^{D-d} y \sqrt{g_{D-d}}\, e^{2A} \left(\frac{\partial g_{ij}}{\partial u^I} \frac{\partial g^{ij}}{\partial u^J} - \frac{1}{D-2} \frac{\partial g}{\partial u^I} \frac{\partial g}{\partial u^J} \right), \tag{4.17}$$

which is a gauge-dependent quantity because in general $\partial_I g_{ij}$ is not orthogonal to gauge transformations.

Starting from G^0_{IJ} we can ask what is the effect of the "compensating gauge transformation"

$$\delta_I g_{ij} = \frac{\partial g_{ij}}{\partial u^I} - \nabla_i \eta_{Ij} - \nabla_j \eta_{Ii} \tag{4.18}$$

[3] We are ignoring the overall factor $M_{P,D}^{D-2}$; also the correct normalization of the d-dimensional Ricci term would introduce a factor of $1/Vol(X)$ in the field space metric.

which projects down to $\mathcal{A}/\mathcal{G}$. More concretely, we are interested in analyzing $G_{IJ} - G^0_{IJ}$, which may be shown to be

$$G_{IJ} - G^0_{IJ} = \frac{1}{4} \int d^{D-d}y \sqrt{g_{D-d}}\, e^{2A}\, \eta_{Ij}\, \nabla_i \left(\frac{\partial \pi^{ij}}{\partial u^J} \right) + (I \leftrightarrow J)\,. \tag{4.19}$$

Let's first derive the explicit projector analogous to the expression Eq. (2.10) for non-abelian Yang-Mills theories. From Eq. (4.13), the compensating fields satisfy the equation

$$\left(g_{ij}\, \nabla^k \nabla_k + 2\nabla_i \nabla_j + R_{ij} \right) \eta^j_I = \nabla^k \left(\frac{\partial g_{ki}}{\partial u^I} \right) \tag{4.20}$$

plus the relation Eq. (4.11) which fixes possible residual gauge transformations preserving Eq. (4.20). Defining the operator

$$\mathcal{O}_{ij} := g_{ij}\, \nabla^k \nabla_k + 2\nabla_i \nabla_j + R_{ij}\,,$$

formally the compensators are given by

$$\eta^i_I = (\mathcal{O}^{-1})^{ij}\, \nabla^k \left(\frac{\partial g_{kj}}{\partial u^I} \right). \tag{4.21}$$

In this way,

$$\delta_I g_{ij} = \frac{\partial g_{ij}}{\partial u^I} - \nabla_i\, (\mathcal{O}^{-1})_{jl}\, \nabla_k \left(\frac{\partial g^{kl}}{\partial u^I} \right) + (i \leftrightarrow j)\,. \tag{4.22}$$

We conclude that the effect of the compensators on the metric is

$$G_{IJ} - G^0_{IJ} = \frac{1}{2} \int d^{D-d}y \sqrt{g_{D-d}}\, e^{2A}\, \nabla_i \left(\frac{\partial g^{ij}}{\partial u^I} \right) \mathcal{O}^{-1}_{jl}\, \nabla_k \left(\frac{\partial g^{kl}}{\partial u^I} \right). \tag{4.23}$$

This is the term responsible for minimizing the metric over each gauge orbit. A different compensator choice would imply that the gauge directions are not projected out, giving a larger result.

5 Application to string compactifications

The Hamiltonian derivation of the field space metric Eq. (4.16) holds quite generally. In particular supersymmetry is not assumed and the details of the matter sector (fluxes, branes, etc.) are not needed.

Of course, given supersymmetry, one can exploit its constraints. For instance, for $\mathcal{N} = 2$ supersymmetries the metric for chiral superfields may be obtained from that of the vector superpartners in the $\mathcal{N} = 2$ multiplet, which enter quadratically in the 10d action. Already for $\mathcal{N} = 1$ susy, deriving the moduli kinetic terms by dimensionally reducing the 10d action supersymmetry is a very involved task, as was shown in [1]. The main obstacle is the correct implementation of the constraints, which arise from the $(0M)$ components of Einstein equations.

On the other hand, we have shown how the kinetic terms arise more naturally from the GR Hamiltonian. In this section, some simple examples of type II compactifications will be analyzed from this point of view.

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5.1 Calabi-Yau manifolds

To gain intuition we begin by discussing Calabi-Yau compactifications, both from the Hamiltonian and Lagrangian viewpoint. An unwarped Calabi-Yau compactification corresponds to

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + g_{ij}(y)dy^i dy^j \,, \tag{5.1}$$

where g_{ij} is a Ricci flat Kähler metric. Holomorphic coordinates are denoted by z^a, $a = 1, 2, 3$, so that the Kähler form is $J = ig_{a\bar{b}}\, dz^a \wedge d\bar{z}^b$. The metric moduli space splits into complex structure deformations $S^\alpha\, \delta_\alpha g_{ab}$, and Kähler deformations $\rho^r\, \delta_r g_{a\bar{b}}$.

The Hamiltonian analysis may be applied straightforwardly to this case. The space-time components of the constraint Eq. (4.7) imply that the metric fluctuations must be traceless, while the internal components tell us that the fluctuations are in harmonic gauge:

$$g^{ij}\, \delta_I g_{ij} = 0 \,,\ \nabla^i(\delta_I g_{ij}) = 0 \,, \tag{5.2}$$

with I running over $(\alpha,\, r)$. These conditions were a choice in the 6d approach of Candelas and de la Ossa [8], but here they emerge as constraints of the 10d Hamiltonian picture. This occurs as follows. Starting from a zero mode $\partial g_{ij}/\partial u^I$ in some arbitrary gauge, the compensators are equivalent to a diffeomorphism transformation $\partial_I g_{ij} \to \delta_I g_{ij} = \partial_I g_{ij} - \nabla_{(i}\, B_{Ij)}$ which point to point imposes the transverse-traceless constraints. The metric Eq. (4.16) gives, after reintroducing the Planck mass,

$$\begin{aligned}
G_{\alpha\bar{\beta}} &= \frac{1}{4V_{CY}} \int d^6y \sqrt{g_6}\, g^{a\bar{c}} g^{b\bar{d}}\, \delta_\alpha g_{ab}\, \delta_\beta g_{\bar{c}\bar{d}} \\
G_{rs} &= \frac{1}{4V_{CY}} \int d^6y \sqrt{g_6}\, g^{a\bar{c}} g^{b\bar{d}}\, \delta_r g_{a\bar{d}}\, \delta_s g_{b\bar{c}} \,.
\end{aligned} \tag{5.3}$$

Let us explain briefly how the zero modes are actually computed, because this will be necessary to understand conformal Calabi-Yau compactifications. Since Eq. (5.1) is a solution without sources, starting from a given background value g^0_{ij}, the zero modes are solutions to

$$R_{ij}(g^0 + \delta g) = 0 \,. \tag{5.4}$$

Recalling the linearized expression for the Ricci tensor [10]

$$\delta R_{ij} = -\frac{1}{2}\nabla^k\nabla_k \delta g_{ij} - \frac{1}{2}\nabla_i\nabla_j \delta g + \nabla^k\nabla_{(i}\delta g_{j)_k} \,,$$

the zero mode fluctuations satisfy

$$-\frac{1}{2}\nabla^k\nabla_k \delta g_{ij} - \frac{1}{2}\nabla_i\nabla_j \delta g + R_{k(ij)l}\delta g^{kl} + \frac{1}{2}\left(\nabla_i\nabla^k \delta g_{kj} + \nabla_j\nabla^k \delta g_{ki}\right) = 0 \,. \tag{5.5}$$

Next, imposing the gauge $\nabla^i \delta g_{ij} = 0$, the trace part can be set to zero and one is left with

$$-\frac{1}{2}\nabla^k\nabla_k \delta g_{ij} + R_{k(ij)l}\delta g^{kl} = 0 \,. \tag{5.6}$$

This gauge-fixed version of $\delta R_{ij} = 0$ is the Lichnerowicz laplacian on Ricci-flat manifolds.[4] On a Kähler manifold the only nonzero components of the Riemann tensor are $R_{a\bar{b}c\bar{d}}$ up to permutations, which implies that the zero modes of mixed ($\delta g_{a\bar{b}}$) and pure (δg_{ab}) type separately verify this equation.

5.2 Conformal Calabi-Yau case

At the next level of complexity, we consider an internal manifold which is a conformal Calabi-Yau, with the conformal factor given by the inverse of the warp factor,

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}(x)\, dx^\mu dx^\nu + e^{-2A(y)}\, \tilde{g}_{ij}(y)\, dy^i dy^j \,, \tag{5.7}$$

where $\tilde{g}_{ij}$ is the CY metric. These type IIb backgrounds preserve $\mathcal{N} = 1$ susy, and the warp factor is generated by BPS sources [3].

In terms of the unwarped fluctuations $\delta_I \tilde{g}_{ij}$, the constraint Eq. (4.11) sets

$$\delta_I A = \frac{1}{8}\, \tilde{g}^{ij} \delta_I \tilde{g}_{ij} \,; \tag{5.8}$$

this fixes the 4d gauge redundancies. Now $\delta\pi_{ij}$ given in Eq. (4.12), becomes the warped harmonic combination

$$\delta_I \pi_{ij} = e^{-2A}\left(\delta_I \tilde{g}_{ij} - \frac{1}{2}\tilde{g}_{ij}\, \delta_I \tilde{g}\right). \tag{5.9}$$

The constraint coming from $D_M \pi^{Mj} = 0$ sets

$$g^{ik}\nabla_i(e^{2A}\delta_I \pi_{kj}) = \tilde{g}^{ik}\tilde{\nabla}_i\left(\delta_I \tilde{g}_{kj} - \frac{1}{2}\tilde{g}_{kj}\, \delta_I \tilde{g}\right) - 4\tilde{g}^{ik}\, \partial_i A\, \delta_I \tilde{g}_{kj} = 0 \,. \tag{5.10}$$

Finally, replacing Eq. (5.9) into the Hamiltonian expression Eq. (4.16), we arrive to the warped moduli space metric

$$G_{IJ}(u) = \frac{1}{4V_W}\int d^6 y \sqrt{\tilde{g}_6}\, e^{-4A}\, \tilde{g}^{ik}\tilde{g}^{jl}\, \delta_I \tilde{g}_{ij}\, \delta_J \tilde{g}_{kl} \,. \tag{5.11}$$

These results agree with those in [1], which were obtained by dimensionally reducing the action. In that approach, the compensators were gauged away; in the Hamiltonian formalism they arise as Lagrange multipliers which can always be set to zero. Furthermore, the rather complicated constraint in the r.h.s. of Eq. (5.10) has a simple interpretation in terms of the full metric with conformal and warp factors, $\nabla^i(e^{2A}\delta_I \pi_{ij}) = 0$. The present derivation suggests that the natural metric fluctuations are $\delta\pi_{ij}$ instead of δA and $\delta\tilde{g}_{ij}$ separately.

The presence of a nontrivial warp factor has important effects on the moduli dynamics. Eq. (5.8) implies that the fluctuations acquire a nonzero trace part proportional to $\delta_I A$; on the other hand, Eq. (5.10) imposes a gauge which is different from the harmonic condition. Therefore, although the fields u^I are the same as in the unwarped case (so that we still have

[4]If the Ricci-tensor doesn't vanish there is an extra term proportional to $R_{ik}\delta g_j{}^k$. However, the Einstein equation would also include a source piece.

complex and Kähler moduli), the internal wavefunctions that support them have changed. From Eq. (4.18), the change is by a diffeomorphism in the underlying CY,

$$\delta_I \tilde{g}_{ij} = \frac{\partial \tilde{g}_{ij}}{\partial u^I} - \tilde{\nabla}_i (e^{2A} \eta_{Ij}) - \tilde{\nabla}_j (e^{2A} \eta_{Ii}) . \tag{5.12}$$

Here $\partial \tilde{g}_{ij}/\partial u^I$ are the unwarped modes from the previous section, which are in transverse traceless gauge. The compensating fields η_{Ii} are then fixed by Eq. (5.8) and Eq. (5.10). The physical zero mode $\delta_I \tilde{g}_{ij}$ is guaranteed to satisfy $\delta \tilde{R}_{ij} = 0$ separately for Kähler and complex deformations; indeed, it differs from the corresponding unwarped mode only by a gauge transformation. Notice however that the zero mode equation is no longer the Lichnerowicz laplacian which is only valid in harmonic gauge. Rather, one would have to solve the full Eq. (5.5). Of course, since we already know $\partial_I \tilde{g}_{ij}$, it is simpler to use the constraints to solve for the compensating fields.

The behavior of the compensators depends on each particular background, but from the discussion of section 4.2 we know that they give a nonzero contribution to the field space metric. In fact, the correct choice will minimize its value on a gauge orbit. One important consequence of this is that the metric Eq. (5.11) could mix complex and Kähler moduli. Indeed, a complex structure fluctuation acquires a nonzero mixed component $\delta_\alpha \tilde{g}_{a\bar{b}}$, while the Kähler moduli also have pure components $\delta_r \tilde{g}_{ab}$. Therefore, there can be mixed terms of the form

$$G_{\alpha r} \sim \frac{1}{V_W} \int d^6 y \sqrt{\tilde{g}_6}\, e^{-4A} \left(\delta_\alpha \tilde{g}_{ab}\, \delta_r \tilde{g}^{ab} + \delta_\alpha \tilde{g}_{a\bar{b}}\, \delta_r \tilde{g}^{a\bar{b}} \right) . \tag{5.13}$$

This can affect KKLT type [11] scenarios including warping, so it would be important to understand better the susy structure of the field space metric.

6 Analysis of the warped deformed conifold

In this last section, the previous formalism is applied to construct the metric of the complex modulus S of the warped deformed conifold. The warp factor is produced by turning on N units of F_3 flux through the A-cycle, and β^{NS} units of H_3 flux through the B-cycle.

Let us first note that this problem has a good supergravity limit, in which α' corrections vanish. One might worry about this point because the unit of flux quantization involves α'. However, one can hold the magnitude of F_3 and H_3 fixed by scaling up the number of flux units as one takes $\alpha' \to 0$. The only remaining dependence on α' is in the ten-dimensional Planck constant, which drops out for $g_s \to 0$. This is the relevant large N limit in gauge/gravity dualities or compactifications with large hierarchies.

For concreteness, consider a coordinate system where the conifold is centered around $r = 0$. At a distance $r \approx \Lambda_0$ the throat is glued to a compact Calabi-Yau along the lines described in [3]. Three regions may then be distinguished:

- $r \geq \Lambda_0$ corresponds to the transition region into the bulk;
- $(g_s N \alpha')^{1/2} \leq r \leq \Lambda_0$ describes a deformed conifold with approximately constant warp factor $e^{-4A} \approx c$;

- $r \ll (g_s N\alpha')^{1/2}$ is the strongly warped limit of the deformed conifold, described by the Klebanov-Strassler solution [12].

Notice that in the large N limit $S \ll \Lambda_0^3$.

In the region $r \geq (g_s N\alpha')^{1/2}$ the warp factor variations may be neglected and the compactification space is a Calabi-Yau manifold. For small S, the bulk contributions are subleading and the metric $G_{S\bar{S}}$ is [13]

$$G_{S\bar{S}} = \frac{k}{V_{CY}} \log \frac{\Lambda_0^3}{|S|} \,. \tag{6.1}$$

Geometrically, the logarithmic dependence follows from a monodromy argument, and from the dual field theory point of view it is related to the running of the gauge coupling [14]. In our present approach, the compensating fields impose the harmonic gauge for metric fluctuations, and the computation of the field space metric is done along the lines of section 5.1.

On the other hand, a very different behavior may be observed in the strongly warped region. In [15] it was conjectured that the field space metric including warp effects is

$$G_{S\bar{S}} = -\frac{\int e^{-4A}\,\chi_S \wedge \bar{\chi}_S}{\int e^{-4A}\,\Omega \wedge \overline{\Omega}} \,. \tag{6.2}$$

Based on this, [2] found a new power-like divergence in the metric,

$$G_{S\bar{S}} = \frac{1}{V_W}\left(c \log \frac{\Lambda_0^3}{|S|} + c' \,\frac{(g_s N\alpha')^2}{|S|^{4/3}} \right) .$$

However, the conjectured form Eq. (6.2) is not orthogonal to gauge transformations since χ_S is harmonic with respect to the unwarped metric, while the physical fluctuations should be harmonic with respect to the full 10d metric.

Our aim is to find the correct metric $G_{S\bar{S}}$ for the strongly warped conifold using the results of section 4 and 5. Before this, we briefly review the KS solution [12].

6.1 The Klebanov-Strassler background

This is the strongly warped limit of the deformed conifold,

$$\sum_a (z^a)^2 = S \,. \tag{6.3}$$

The full 10d metric reads [12]

$$\begin{aligned} ds_{10}^2 = {} & \frac{|S|^{2/3}}{2^{1/3}(g_s N\alpha')} I(\tau)^{-1/2}\, \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{2^{2/3}} (g_s N\alpha')\, I(\tau)^{1/2} K(\tau) \times \\ & \times \left[\frac{1}{3K(\tau)^3} \left(d\tau^2 + (g^5)^2 \right) + \cosh^2\left(\frac{\tau}{2}\right) \left((g^3)^2 + (g^4)^2 \right) + \sinh^2\left(\frac{\tau}{2}\right) \left((g^1)^2 + (g^2)^2 \right) \right] \end{aligned} \tag{6.4}$$

and the warp factor is given by

$$e^{-4A(\tau)} = 2^{2/3}\, \frac{(g_s N\alpha')^2}{|S|^{4/3}}\, I(\tau) \,. \tag{6.5}$$

The model is regularized in terms of the UV cutoff τ_Λ defined by $e^{-4A(\tau_\Lambda)} \approx 1$.

A very interesting feature of this solution is that the warped 6d metric becomes independent of the complex modulus S, which only enters in the redshift factor of the observable energy. This is due to the fact that in the noncompact limit the S-dependence from the warp factor cancels that of the unwarped metric. As a result, the energy scales of fluctuations localized in the throat are essentially controlled by the minimum redshift

$$e^{A_{min}} \sim |S_{min}|^{1/3} = \Lambda \,.$$

In the dual gauge theory this is the statement that there is a mass gap given by the dynamical scale Λ.

From this viewpoint, it is not easy to interpret geometric quantities such as $\int e^{-4A} \chi_S \wedge \bar{\chi}_S$, given that the warped internal metric does not vary under a complex deformation. Therefore, let us explain how the metric for the S-field arises. In this case $\partial_S g_{ij} = 0$, so it is better to work directly with the original expression Eq. (4.16),

$$G_{S\bar{S}} = \frac{1}{4V_W} \int d^6y \sqrt{g_6}\, e^{2A}\, g^{ik} g^{jl} \delta_{\bar{S}} g_{ij}\, \delta_S \pi_{kl} \,. \tag{6.6}$$

Since $\partial_S g_{ij} = 0$, we have (suppressing the subindex 'S' in η_{Si})

$$\delta_S g_{ij} = -\nabla_i \eta_j - \nabla_j \eta_i \,.$$

Hence the internal metric fluctuation is produced solely by the compensating field! This contribution is nonzero because a time-dependent fluctuation in S does modify the 4d piece of the metric, and this requires non-vanishing compensators. Thus the KS solution is very good for illustrating the effects of compensators, since $G_{S\bar{S}}$ would vanish if they were not taken into account.

Plugging this metric fluctuation into Eq. (6.6), the integrand becomes a total derivative. Integrating over τ gives

$$G_{S\bar{S}} = -\frac{1}{2V_W} \left(\int \prod_i g^i \right) \sqrt{g_6}\, e^{2A}\, \eta_i\, \delta_S \pi^{i\tau} \Big|_{\tau=0}^{\tau=\tau_\Lambda} \,. \tag{6.7}$$

In the remaining of the section we will compute this quantity. Now we turn to finding the compensating fields, from which the fluctuation $\delta_S \pi_{ij}$ will be obtained (see Eq. (4.12)).

6.2 Compensating fields

Solving the compensator equations explicitly is a very involved task. Indeed, Eq. (4.7) (or, equivalently, Eq. (4.11) and Eq. (4.14)) gives a system of six coupled second order PDEs, with coefficients that contain various combinations of (hyperbolic) trigonometric functions, plus $I(\tau)$ which only has an integral expression. Now, the problem is simplified by the fact that in order to evaluate Eq. (6.7) only the solutions close to the boundaries are needed. The approach is then to expand the KS solution near each boundary, and find the solutions separately in each region after making simplifying ansatze for the compensators taking into account the isometries of the background.

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Still the problem turns out to be too complicated to allow for an intuitive understanding of the underlying physics. Instead, we will consider the so-called *hard-wall* approximation, where the regular background is replaced by an AdS space with a cut-off at $r = |S|^{1/3}$ plus boundary conditions to match the known KS values. The warp factor is taken to be

$$e^{-4A(r)} = \frac{a_0(g_s N\alpha')^2}{r^4} \tag{6.8}$$

where $a_0 = 2^{2/3}\, I(0)$ is chosen so that at $r = |S|^{1/3}$ this agrees with the KS warp factor at $\tau = 0$. Similarly, the 10d metric will be approximated by

$$ds^2_{10} = e^{2A(r)}\, \eta_{\mu\nu}\, dx^\mu dx^\nu + e^{-2A(r)} \big(dr^2 + r^2\, ds^2_{T^{1,1}}\big)\,. \tag{6.9}$$

In the hard-wall approximation there is one IR boundary at $r = |S|^{1/3}$ and the space has a UV cutoff at $r = \Lambda_0$. However, due to the fall-off of the metric fluctuations at large r, only the IR boundary turns out to contribute to the field space metric. Therefore we only need to solve for the compensators around the tip of the conifold.

Before proceeding, let us pause and ask about the validity of this approximation. The work of [16] performed a detailed numerical analysis of the mass spectrum in the full KS solution without any approximation in the background. Their results were compared to the ones obtained in the hard-wall approximation and it is found that, although the precise numerical coefficients don't agree, both spectra have the same dependence on the parameters of the problem. Since the masses depend directly on the kinetic term metric, the hard-wall method gives the correct dependence on $g_s N\alpha'$ and S, while more work would be required to get the numerical coefficients right.

From Eq. (4.11) and Eq. (4.14), the constraint equations that need to be solved are

$$\begin{aligned} g^{ij}\, \nabla_i\, \eta_j + 2\, g^{ij}\, \partial_i A\, \eta_j \,&=\, 2\, \partial_S A \\ g^{ij}\, \nabla_i(\delta_S \pi_{jk}) + 2\, g^{ij}\, \partial_i A\, \delta_S \pi_{jk} \,&=\, 0 \end{aligned} \tag{6.10}$$

with

$$\delta_S \pi_{ij} = -\nabla_i \eta_j - \nabla_j \eta_i - g_{ij}\, (g^{kl}\, \nabla_k \eta_l)\,.$$

The covariant derivatives here are with respect to the warped 6d metric g_{ij}.

Due to the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ symmetry, the angular components of the compensators may be rotated to point in the ψ direction. A radial compensator is of course needed due to the source term produced by $\partial_r A$. Then from Eq. (6.10) we learn that η_r and η_ψ only depend on the radial direction. Notice that at least two nonzero components are needed to be able to construct a metric fluctuation orthogonal to gauge transformations. Summarizing, our ansatz for the compensating field is

$$\eta_i(y) = \big(\eta_r(r),\, \eta_\psi(r), 0, 0, 0, 0\big) \tag{6.11}$$

where the last 4 components refer to the coordinates (θ_i, ϕ_i).

This is admittedly not the most general ansatz; one could find others with less symmetry. However, since the kinetic term coefficient Eq. (6.6) is the integral of a positive definite quantity, it seems very implausible to us that a solution with less symmetry could lead to a smaller result.

Granting Eq. (6.11), the system Eq. (6.10) then becomes one second order equation for η_ψ and two equations (one first order and one second order) for η_r. Concentrating on η_r first, the general solution to the first order equation is

$$\eta_r(r) = \sqrt{a_0}\,\frac{(g_s N\alpha')}{|S|}\,\frac{1}{r} + \frac{c_1}{r^3}$$

Plugging this into the second order constraint sets $c_1 = 0$. The role of this compensator is to cancel the contribution of the nontrivial warp factor; it may be checked that η_r is covariantly constant, $\nabla_r \eta_r = 0$. This then implies that

$$g^{kl}\nabla_k \eta_l = 0\,,\ \ \delta_S \pi_{rr} = 0\,.$$

Due to these properties, η_r drops out from the second order equation for η_ψ, and the solution around $r \approx |S|^{1/3}$ is

$$\eta_\psi(r) = \frac{b_1}{r}\,.$$

The constant b_1 is fixed by matching $||\delta_S \pi_{\psi r}||^2$ at $r = |S|^{1/3}$ to $||\chi_S||^2$ at $\tau = 0$, ensuring that the metric fluctuations are normalized in the same way. This boundary condition is required because the IR cutoff $r = |S|^{1/3}$ is imposed by hand. The result is

$$\eta_\psi(r) \approx k\,\frac{(g_s N\alpha')}{|S|^{2/3}}\,\frac{1}{r}\,,$$

where from now on we will absorb the dimensionless order one constants into k. The dependence on $(g_s N\alpha')$ and $|S|^{2/3}$ can also be understood as follows. Since $\delta g_{\psi r} = e^{-2A}\,\delta\tilde{g}_{\psi r}$ and $\delta\tilde{g}$ is independent of fluxes, the warped metric fluctuation has to be proportional to $(g_s N\alpha')$. Then $|S|^{-2/3}$ follows from dimensional analysis.

Putting these results together, the compensating field in the hard-wall approximation is

$$\eta_i(y) = \left(\sqrt{a_0}\,\frac{(g_s N\alpha')}{|S|}\,\frac{1}{r},\ k\,\frac{(g_s N\alpha')}{|S|^{2/3}}\,\frac{1}{r}, 0, 0, 0, 0\right). \qquad (6.12)$$

With these components, the only nonvanishing metric fluctuation is

$$\delta_S \pi_{\psi r} = -k\,\frac{(g_s N\alpha')}{|S|^{2/3}}\,\frac{1}{r^2}\,. \qquad (6.13)$$

Naively, one might find it peculiar that the metric variation is an off-diagonal component, not present in the original Klebanov-Strassler metric Eq. (6.4). But, as we commented, the 6d part of the Klebanov-Strassler metric is actually independent of S, and the variation is pure gauge. Nevertheless it must be non-zero to satisfy the orthogonality condition.

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6.3 Metric including compensator effects

To compute the field space metric we need to replace Eq. (6.12) into the expression Eq. (6.7),

$$G_{S\bar{S}} = -\frac{\mathrm{vol}(T^{1,1})}{2V_W}\, k^2\, r^5\, e^{-4A}\, g^{\psi\psi}\, g^{rr}\, \eta_\psi\, \delta_S \pi_{\psi r}\,,$$

and then evaluate this at $r = |S|^{1/3}$. The result is

$$G_{S\bar{S}} = k\, \frac{\mathrm{vol}(T^{1,1})}{V_W}\, \frac{(g_s N \alpha')^2}{|S|^{4/3}}\,, \tag{6.14}$$

where we have combined all the order one numerical constants into k. This metric agrees qualitatively with the one found by [2].

We have arrived to the same functional dependence on S but through a very different path, by requiring orthogonality with respect to gauge transformations in the presence of warp and conformal factors. It is thus instructive to connect our results to the expression Eq. (6.2) in terms of the $(2,1)$ form χ_S.

First, the effect of the η_r compensator is simply to set

$$\delta_S A = 0\,,\ \delta_S g = 0\,.$$

In terms of the physical fluctuations, the warp factor becomes independent of S and the metric fluctuation is traceless. In fact, both are equivalent by the constraint Eq. (5.8). Then the other constraint (Eq. (5.10)) may be rewritten as

$$\tilde{\nabla}^i \big(e^{-4A}\, \delta_S \tilde{g}_{ij}\big) = 0 \tag{6.15}$$

which is a warped generalization of the harmonic gauge. The associated 3-form

$$\chi_S = \tilde{g}^{ln}\, \Omega_{ijl}\, \delta_S \tilde{g}_{nk}\, dy^i dy^j dy^k \tag{6.16}$$

then satisfies

$$d \star_6 \left(e^{-4A}\, \chi_S\right) = 0\,. \tag{6.17}$$

In other words, the effect of η_ψ is to shift the original harmonic $(2,1)$ form by an exact piece so that the "physical" χ_S satisfies Eq. (6.17).

With this constraint, the field space metric reads

$$G_{S\bar{S}} = -\frac{\int e^{-4A}\, \chi_S \wedge \star_6\, \bar{\chi}_S}{\int e^{-4A}\, \Omega \wedge \overline{\Omega}}\,. \tag{6.18}$$

The Hodge star is needed because χ_S is no longer harmonic.

After having established this, it becomes clearer why we find the same $1/|S|^{4/3}$ behavior as in [2]. The reason is that the original harmonic form is shifted by an exact piece in order to satisfy Eq. (6.17), but in the KS coordinates this equation is independent of S. Hence neither the $(2,1)$ form nor the exact 3-form add extra S dependence to Eq. (6.18). In fact all of the S dependence comes from the warp factor, which is proportional to $|S|^{-4/3}$. This can be extracted, and the remaining integral leads to an order one coefficient. As the integrand is different, its numerical value is probably different than that of [2]. But since the correct field space metric minimizes a positive definite inner product, the result must be equal or smaller than that found in [2].

The upshot is that the expression Eq. (6.2) was qualitatively correct in this case, however it is not yet clear in what generality this is true as the argument we just gave depends on special properties of the KS solution.

To conclude, we would like to point out that, while our approach does not use supersymmetry, it would be important to understand which are the implications of these results for the 4d Kähler potential. For instance, while we have proved that Eq. (6.17) holds for the conifold, this may also be valid in compactifications which admit a covariantly constant spinor in six dimensions. Another possible application is to computing kinetic terms from compactifications which are not conformally equivalent to Calabi-Yau manifolds. Such backgrounds may describe gravity duals of metastable vacua in strongly coupled gauge theories; see [17] for a recent example. We plan to come back to this in the future.

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